Evolution of space curves and the Pohlmeyer-Lund-Regge equation: explicit solutions via Riemann theta functions

Yuhei Kogo
partly joint work with Shimpei Kobayashi and Nozomu Matsuura

Hokkaido University

August 29, 2025

第 72 回幾何学シンポジウム
筑波大学

This work was supported by JST SPRING, Grant Number JPMJSP2119.

Introduction

Space curve evolution appears in geometry, fluid dynamics, and integrable systems.
Vortex filament motion (1972):

$$\dot{\gamma} = \gamma'' \times \gamma', \quad \text{(binormal motion)}$$

Hasimoto's transform: $q = \kappa e^{i \int \tau ds} \Rightarrow$ nonlinear Schrödinger eq.

$$i\dot{q} + q'' + 2|q|^2 q = 0$$

Hasimoto H. A soliton on a vortex filament. Journal of Fluid Mechanics. 1972

- Soliton and quasi-periodic solutions of nonlinear Schrödinger eq.

  E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev.

  Algebro-Geometric Approach to Nonlinear Integrable Equations. *Springe* 1994.
- Differential geometric properties.
  Calini, A., Ivey, T. Finite-Gap Solutions of the Vortex Filament Equation
  : Genus One Solutions and Symmetric Solutions. *J Nonlinear Sci.* (2005)

  : Isoperiodic Deformations. *J Nonlinear Sci.* (2007)

Lund-Regge evolution (1978):

$$\dot{\gamma}' = \gamma' \times \dot{\gamma} \Rightarrow \text{PLR equation for curvature/torsion}$$

F. Lund, T. Regge. Unified approach to strings and vortices with soliton solutions. *Phys. Rev.* (1978)

K. Pohlmeyer. Integrable Hamiltonian systems and interactions through quadratic constraints. *Comm. Math. Phys.* (1976)

- Soliton and quasi-periodic solutions of PLR eq.

  E. Date. Multi-soliton solutions and quasi-periodic solutions of nonlinear equations of sine-Gordon type. *Osaka J. Math.* (1982)

- Differential geometric properties.

  C. Chen, Y. Li. The Lund-Regge surface and its motion's evolution equation. *J. Math. Phys.* (2002)

  Kobayashi, S., K. and Matsuura, N. The Evolution of a Curve Induced by the Pohlmeyer-Lund-Regge Equation. *J Nonlinear Sci.* (2025)

Our goal:

- Reformulate the Lund-Regge evolution with a 2×2 SU(2) Frenet frame.
- Encode curvature-torsion data in a Hasimoto-type field q and link it to the PLR equation.
- Produce explicit, quasi-periodic solutions via Riemann θ-functions and reconstruct the curve evolution.

The Lund-Regge evolution

Definition 1

A space curve evolution γ will be called the $\mathrm{Lund\text{-}Regge\ evolution}$ if the following equation holds:

$$\dot{\gamma}' = \gamma' \times \dot{\gamma}.$$

Furthermore, the Lund-Regge evolution will be called $\mathrm{regular}$ if $\gamma' \times \dot{\gamma} \neq 0$ holds.

Remark 0.1

In the original definition by Lund and Regge [LundRegge78], the following constraints were also imposed:

$$|\gamma'| = \ell, \quad |\dot{\gamma}| = \ell^{-1}, \quad \ell > 0.$$

However, these additional conditions are not essential.

Theorem 2 (Kobayashi, K, Matsuura25)

Regarding the Lund-Regge evolution, we have the following:

(1) *Let γ be the Lund-Regge evolution such that $|\gamma'| = 1$. Then it satisfies*

$$\dot{\gamma} = -\int^s \dot{\kappa}\kappa \, ds T - \dot{\kappa} N - \kappa \int^s \dot{\tau} ds B,$$

and its Frenet frame $\mathcal{F}$ evolves according to following

$$\mathcal{F}' = \mathcal{F}\mathcal{L}, \text{ where } \mathcal{L} = \frac{1}{2}\begin{pmatrix} i\tau & -\kappa \\ \kappa & -i\tau \end{pmatrix}, \tag{1}$$

$$\dot{\mathcal{F}} = \mathcal{F}\mathcal{M}, \text{ where } \mathcal{M} = \frac{1}{2}\begin{pmatrix} \frac{i}{\kappa}\left(-\dot{\kappa}' + \kappa \int^s \dot{\tau} ds \tau\right) & -\kappa \int^s \dot{\tau} ds + i\dot{\kappa} \\ \kappa \int^s \dot{\tau} ds + i\dot{\kappa} & -\frac{i}{\kappa}\left(-\dot{\kappa}' + \kappa \int^s \dot{\tau} ds \tau\right) \end{pmatrix}.$$

Moreover, the curvature κ and torsion τ of γ satisfy the pair of PDEs

$$\frac{\dot{\kappa}'}{\kappa} - (\tau - 1)\int^s \dot{\tau} ds + \int^s \kappa\dot{\kappa} ds = 0, \quad \left(\kappa \int^s \dot{\tau} ds\right)' + \dot{\kappa}(\tau - 1) = 0,$$

which is the compatibility condition between (1).

Theorem 3 (Kobayashi, K, Matsuura25)

(2) *Conversely, let* κ *and* τ *be solutions to the system*

$$\frac{\dot{\kappa}'}{\kappa} - (\tau - 1) \int^s \dot{\tau} ds + \int^s \kappa \dot{\kappa} ds = 0, \quad \left(\kappa \int^s \dot{\tau} ds \right)' + \dot{\kappa}(\tau - 1) = 0,$$

and let γ *be the evolution determined by*

$$\dot{\gamma} = - \int^s \dot{\kappa} \kappa \, ds T - \dot{\kappa} N - \kappa \int^s \dot{\tau} ds B,$$

where the integration and derivative are taken with respect to the length element. Then $|\gamma'|$ *does not depend on* t *and thus* γ *gives an isoperimetric curve flow. Moreover, by choosing the arc-length parametrization* $|\gamma'| = 1$ *of* γ*, the curve* γ *satisfies the Lund-Regge evolution* $\dot{\gamma}' = \gamma' \times \dot{\gamma}$ *with unit-speed.*

Corollary 4 (Kobayashi, K, Matsuura25)

By choosing the diagonal gauge $\mathcal{D} = \mathrm{diag}(\exp(-ip), \exp(ip))$, *where* $p = \frac{1}{2}\int^s (\tau - 1)ds - \frac{\pi}{2}$ *and introducing the complex-valued function*

$$q = \kappa \exp\left(i \int^s (\tau - 1)ds\right),$$

the evolution of the gauged Frenet frame $F = \mathcal{F}\mathcal{D}$ *can be rephrased as*

$$F' = FL, \quad L = \frac{1}{2}\begin{pmatrix} i & q \\ -\bar{q} & -i \end{pmatrix},$$
$$\dot{F} = FM, \quad M = \frac{i}{2}\begin{pmatrix} -\mathrm{Re}\,(\dot{q}'/q) & -\dot{q} \\ -\dot{\bar{q}} & \mathrm{Re}\,(\dot{q}'/q) \end{pmatrix}.$$

The compatibility condition of the above system can be computed as $\{\mathrm{Re}\,(\dot{q}'/q)\}' = -\frac{1}{2}\left(|q|^2\right)^{\cdot}$, *and* $\mathrm{Im}\,(\dot{q}'/q) = 0$, *which are equivalent to the following nonlinear PDE:*

$$\dot{q}' + \frac{1}{2}q\int^s \left(|q|^2\right)^{\cdot} ds = 0.$$

Corollary 5 (Kobayashi, K, Matsuura25)

There exists a family $\{F^\lambda\}_{\lambda>0}$ *such that* F^λ *satisfies the following system of PDEs:*

$$(F^\lambda)' = F^\lambda L^\lambda, \quad L^\lambda = \frac{1}{2}\begin{pmatrix} i\lambda & q \\ -\bar{q} & -i\lambda \end{pmatrix},$$

$$(F^\lambda)^{\cdot} = F^\lambda M^\lambda, \quad M^\lambda = \frac{i}{2\lambda}\begin{pmatrix} -\operatorname{Re}(\dot{q}'/q) & -\dot{q} \\ -\dot{\bar{q}} & \operatorname{Re}(\dot{q}'/q) \end{pmatrix}.$$

Recall that the Euclidean three-space $\mathbb{R}^3$ can be identified with $\mathfrak{su}(2)$ by

$$\begin{pmatrix} p & q & r \end{pmatrix}^{\mathsf{T}} \in \mathbb{R}^3 \longleftrightarrow \frac{1}{2}\begin{pmatrix} ir & -p-iq \\ p-iq & -ir \end{pmatrix} \in \mathfrak{su}(2).$$

Theorem 6 (Kobayashi, K, Matsuura25)

Let q *be a solution of* $\dot{q}' + \frac{1}{2}q\int^s \left(|q|^2\right)^{\cdot} ds = 0$, *and* $F(= F^\lambda)$ *the solution of the Lax pair*

$$(F^\lambda)' = F^\lambda L^\lambda, \quad L^\lambda = \frac{1}{2}\begin{pmatrix} i\lambda & q \\ -\bar{q} & -i\lambda \end{pmatrix},$$

$$(F^\lambda)^{\cdot} = F^\lambda M^\lambda, \quad M^\lambda = \frac{i}{2\lambda}\begin{pmatrix} -\operatorname{Re}(\dot{q}'/q) & -\dot{q} \\ -\dot{\bar{q}} & \operatorname{Re}(\dot{q}'/q) \end{pmatrix}.$$

Define an $\mathfrak{su}(2)$*-valued map*

$$\gamma = \lambda(\partial_\lambda F)F^{-1}|_{\lambda=1},$$

where $\partial_\lambda = \frac{\partial}{\partial\lambda}$. *Then under the identification* $\mathbb{R}^3 \cong \mathfrak{su}(2)$, γ *is the Lund-Regge evolution* $\dot{\gamma}' = \gamma' \times \dot{\gamma}$ *with* $|\gamma'| = 1$. *Conversely, all Lund-Regge evolutions can be obtained by this way.*

From the foregoing discussion, an explicit description of the Lund-Regge evolution requires constructing a solution—i.e., a wave function of the Lax pair

$$
\begin{aligned}
(F^\lambda)' &= F^\lambda L^\lambda, \quad L^\lambda = \frac{1}{2}\begin{pmatrix} i\lambda & q \\ -\bar{q} & -i\lambda \end{pmatrix}, \\
(F^\lambda)^{\cdot} &= F^\lambda M^\lambda, \quad M^\lambda = \frac{i}{2\lambda}\begin{pmatrix} -\operatorname{Re}(\dot{q}'/q) & -\dot{q} \\ -\dot{\bar{q}} & \operatorname{Re}(\dot{q}'/q) \end{pmatrix},
\end{aligned}
$$

While several approaches to building such wave functions are known, we next introduce solutions of the following two types:

1. N-soliton solutions given by the Date Direct method [Date82]
2. Quasi-periodic solutions in terms of Riemann theta function

Date's N-Soliton Solutions

We consider (2×2)-matrix-valued function $\Psi(s,t,\lambda) = (\Psi_{jk}(s,t,\lambda))$ of the following forms

$$\Psi_{11}(s,t,\lambda) = (\lambda^N + \sum_{j=0}^{N-1} \psi_{1j}(s,t)\lambda^j) \exp(2^{-1} i(\lambda s + \lambda^{-1} t)),$$
$$\Psi_{21}(s,t,\lambda) = -(\sum_{j=0}^{N-1} \psi_{2j}(s,t)\lambda^j) \exp(-2^{-1} i(\lambda s + \lambda^{-1} t)),$$
$$\Psi_{12}(s,t,\lambda) = -\overline{\Psi_{12}(s,t,\bar{\lambda})}, \quad \Psi_{22}(s,t,\lambda) = \overline{\Psi_{11}(s,t,\bar{\lambda})}.$$

Let α_j be mutually distinct complex numbers such that for all j $\operatorname{Im}\alpha_j$ have same signature and c_j be arbitrary complex numbers,then we determine $\psi_{ij}(s,t)$ by the following system of linear equations

$$\begin{pmatrix} EA & -CE^{-1}A \\ \overline{CE^{-1}A} & \overline{EA} \end{pmatrix} {}^t(\psi_{10}, \cdots, \psi_{1,N-1}, \psi_{20}, \cdots, \psi_{2,N-1})$$
$$= -{}^t(\alpha_1^N e(\alpha_1), \cdots, \alpha_N^N e(\alpha_N), \overline{c_1 \alpha_1^N} e(\bar{\alpha}_1), \cdots, \overline{c_N \alpha_N^N} e(\bar{\alpha}_N))$$

where A is the $(N \times N)$matrix with (j,k)-elements α_j^{k-1} and E, C are diagonal matrices of order N with entries $e(\alpha_j), c_j$,respectively and $e(\lambda) = \exp(2^{-1} i(\lambda s + \lambda^{-1} t))$.

From [Date82, section 3], we can see that the function $\Psi(s,t,\lambda)$ is a wave function of the following Lax pair

$$\Psi^{-1}\Psi' = \frac{1}{2}\begin{pmatrix} i\lambda & q \\ -\bar{q} & -i\lambda \end{pmatrix}, \quad \Psi^{-1}\dot{\Psi} = \frac{i}{2\lambda}\begin{pmatrix} -\operatorname{Re}(\dot{q}'/q) & -\dot{q} \\ -\dot{\bar{q}} & \operatorname{Re}(\dot{q}'/q) \end{pmatrix},$$

where $q = i\bar{\psi}_{2,N-1}$.

Proposition 0.2 (Kobayashi, K, Matsuura25)

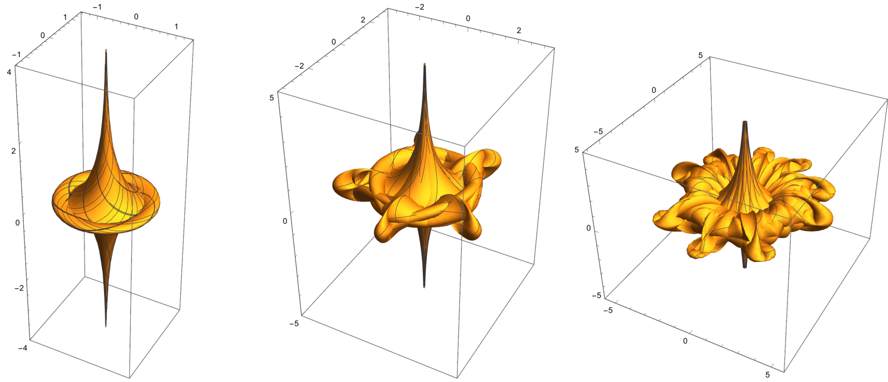
The explicit formula for the N-soliton curves $\gamma(s,t) = (\gamma_1(s,t), \gamma_2(s,t), \gamma_3(s,t))^{\mathsf{T}}$ is given as follows:

$$\gamma_1(s,t) = -2\operatorname{Im}\left[\frac{\bar{f}\partial_\lambda g - g\partial_\lambda \bar{f}}{|f|^2+|g|^2}e^{-2ik}\right]\Bigg|_{\lambda=1},$$

$$\gamma_2(s,t) = 2\operatorname{Re}\left[\frac{\bar{f}\partial_\lambda g - g\partial_\lambda \bar{f}}{|f|^2+|g|^2}e^{-2ik}\right]\Bigg|_{\lambda=1},$$

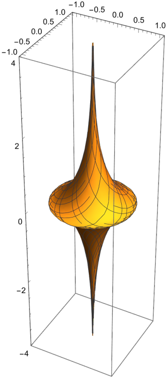
$$\gamma_3(s,t) = s - t + 2\operatorname{Im}\left[\frac{\bar{f}\partial_\lambda f - g\partial_\lambda \bar{g}}{|f|^2+|g|^2}\right]\Bigg|_{\lambda=1}.$$

where $k(\lambda) = (\lambda s + \lambda^{-1}t)/2, f(\lambda) = \lambda^N + \sum_{j=0}^{N-1}\psi_{1j}(s,t)\lambda^j, g(\lambda) = -(\sum_{j=0}^{N-1}\psi_{2j}(s,t)\lambda^j)$

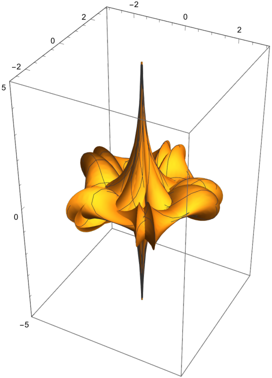


(a) PLR 1-soliton (b) PLR 2-soliton (c) PLR 3-soliton

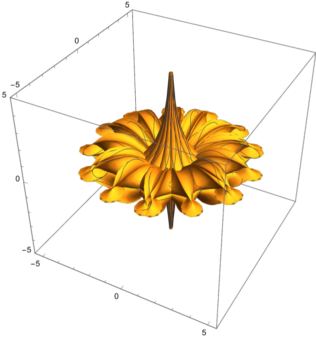
Figure: Swept surfaces formed by the curves with $1, 2,$ and 3-soliton solutions for the PLR equations.



(a) s-G 1-soliton

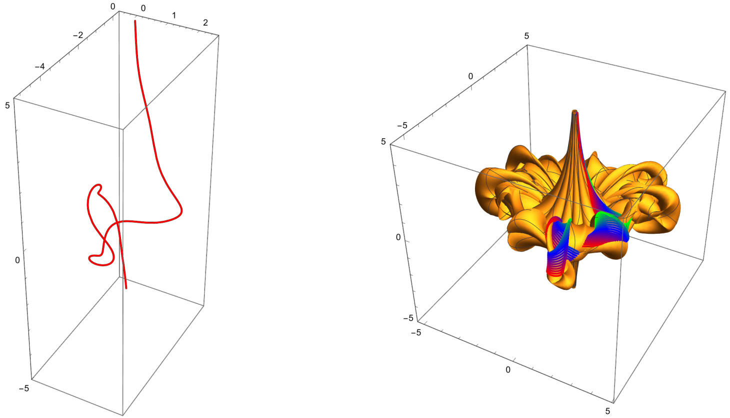


(b) s-G 2-soliton



(c) s-G 3-soliton

Figure: Swept surfaces formed by the curves with $1, 2,$ and 3-soliton solutions for the sine-Gordon equations.



(a) A space curve at time $t = 0$

(b) The curves of t=0,1,2 and its swept surface

Figure: A curve evolution determined from a 4-soliton solution of the PLR equation and its swept surface by the curves.

Hyperelliptic Riemann surface

Let $\mathcal{R}$ be a genus-g hyperelliptic Riemann surface defined by

$$\mu^2 = \prod_{j=1}^{g+1}(\lambda - \lambda_j)(\lambda - \bar{\lambda}_j), \quad \lambda_j \neq \lambda_k (j \neq k),\ \lambda_j \neq \bar{\lambda}_k, \lambda_j \neq 0.$$

Let $P_\infty^\pm$ and $P_0^\pm$ be the points over $\lambda = \infty$ and 0, with local parameters $z = \lambda^{-1}$ near $P_\infty^\pm$, and λ near $P_0^\pm$.

Anti-holomorphic Involution and Homology Basis

There exists a fixed-point-free anti-holomorphic involution

$$\sigma : (\mu, \lambda) \mapsto (-\bar{\mu}, \bar{\lambda})$$

on the Riemann surface $\mathcal{R}$.
Let $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ be a homology basis of $\mathcal{R}$, satisfying

$$a_i \cdot a_j = b_i \cdot b_j = 0, \quad a_i \cdot b_j = \delta_{ij}.$$

We choose a canonical basis such that

$$\begin{aligned} \sigma(a_j) &= a_j, \\ \sigma(b_j) &= -b_j + \sum_{k \neq j} a_k, \quad j = 1, \dots, g. \end{aligned}$$

This homology basis is illustrated in the following figure.

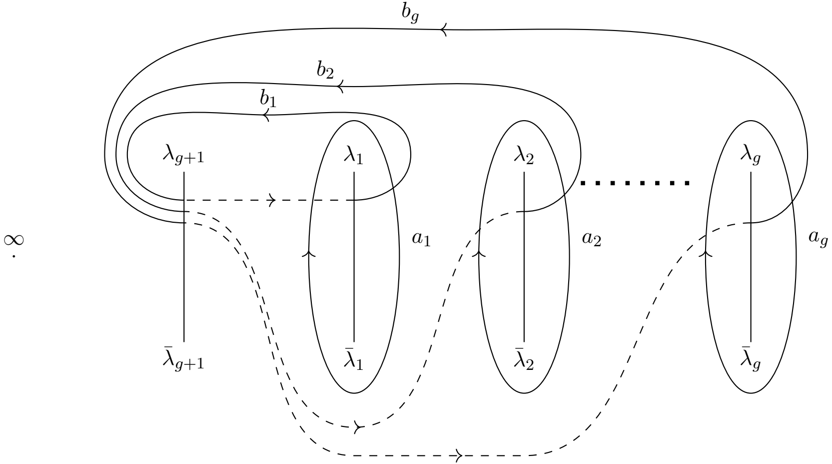


Figure: Homology basis for hyperelliptic Riemann surface $\mathcal{R}$

Let $\omega_1, \ldots, \omega_g$ be holomorphic differentials on $\mathcal{R}$, normalized by

$$\int_{a_k} \omega_j = 2\pi i\, \delta_{kj}, \quad j, k = 1, \ldots, g.$$

Define the period matrix $\tau = (\tau_{jk})$ by

$$\tau_{jk} = \int_{b_k} \omega_j.$$

The associated Riemann theta function is given by

$$\theta(\mathbf{u}) = \sum_{\mathbf{n} \in \mathbb{Z}^g} \exp\left(\tfrac{1}{2}\mathbf{n}\tau{}^t\mathbf{n} + \mathbf{n}{}^t\mathbf{u}\right).$$

The series is absolutely convergent; this follows from the fact that $\operatorname{Re}\tau$ is a negative-definite matrix. Moreover, if $\mathbf{e}_k$ are the standard basis vectors of $\mathbb{C}^g$ and $\boldsymbol{\tau}_k = \tau\mathbf{e}_k$, for $k = 1, \ldots, g$, then

$$\theta(\mathbf{u} + 2\pi i\mathbf{e}_k) = \theta(\mathbf{u}), \quad \theta(\mathbf{u} + \boldsymbol{\tau}_k) = \exp(-\frac{1}{2}\tau_{kk} - u_k)\theta(\mathbf{u}).$$

Under this choice of homology basis, the period matrix τ satisfies the following property:

$$\operatorname{Im} \tau_{jk} = \begin{cases} \pi & \text{if } (j,k) \neq (\ell,\ell), \\ 0 & \text{if } (j,k) = (\ell,\ell). \end{cases}$$

Moreover, the Riemann theta function satisfies the conjugation symmetry:

$$\overline{\theta(\mathbf{u})} = \theta(\bar{\mathbf{u}}).$$

Let Γ be the lattice in $\mathbf{C}^g$ generated by the columns of the matrix $(2\pi i I \mid \tau)$. The Jacobian variety of $\mathcal{R}$ is defined as

$$\operatorname{Jac}(\mathcal{R}) = \mathbf{C}^g/\Gamma.$$

The map

$$\mathcal{A}_Q(P) = \left(\int_Q^P \omega_1, \ldots, \int_Q^P \omega_g \right) \bmod \Gamma,$$

is called the **Abel-Jacobi map**.

Let Ω_j and $d\Omega_j$ $(j = 1, 2, 3)$ be the normalized Abelian integrals and differentials defined by:

$$\Omega_1(P) = \int_{\bar{\lambda}_{g+1}}^{P} d\Omega_1 \sim \pm\left(\frac{1}{z} - \frac{E}{2}\right) \qquad \text{as } P \to P_\infty^{\pm},$$

$$\Omega_2(P) = \int_{\bar{\lambda}_{g+1}}^{P} d\Omega_2 \sim \pm\left(\frac{1}{\lambda} - \frac{F}{2}\right) \qquad \text{as } P \to P_0^{\pm},$$

$$\Omega_3(P) = \int_{\bar{\lambda}_{g+1}}^{P} d\Omega_3 \sim \mp\left(\log z - \tfrac{\pi i}{2} + \tfrac{1}{2}\log\beta\right) \quad \text{as } P \to P_\infty^{\pm}, \quad \beta > 0.$$

Let the b-periods of these differentials be

$$U_j = \int_{b_j} d\Omega_1, \quad V_j = \int_{b_j} d\Omega_2, \quad r_j = \int_{b_j} d\Omega_3 = \int_{P_\infty^-}^{P_\infty^+} \omega_j, \quad j = 1, \ldots, g,$$

and define the vectors $\mathbf{U} = (U_1, \ldots, U_g)$, $\mathbf{V} = (V_1, \ldots, V_g)$, $\mathbf{r} = (r_1, \ldots, r_g)$.
Then the wave function can be constructed as follows.

Proposition 0.3

The wave function in terms of Riemann theta function

$$\Psi = \frac{1}{\sqrt{|\psi_1|^2 + |\psi_2|^2}} \begin{pmatrix} \psi_1 & \psi_2 \\ -\bar{\psi}_2 & \bar{\psi}_1 \end{pmatrix}.$$

is given by:

$$\psi_1 = -i \exp\left[\tfrac{is}{2}(\Omega_1 + \tfrac{E}{2}) + \tfrac{it}{2}(\Omega_2 - \tfrac{H}{2}) + \Omega_3\right] \frac{\theta(\mathcal{A}_-(P) - \mathbf{W} - \mathbf{D} - \mathbf{r})}{\theta(\mathbf{W} + \mathbf{D})},$$

$$\psi_2 = \exp\left[\tfrac{is}{2}(\Omega_1 - \tfrac{E}{2}) + \tfrac{it}{2}(\Omega_2 + \tfrac{H}{2})\right] \frac{\theta(\mathcal{A}_-(P) - \mathbf{W} - \mathbf{D})}{\theta(\mathbf{W} + \mathbf{D})}.$$

where $\mathbf{W} = -\frac{is}{2}\mathbf{U} - \frac{it}{2}\mathbf{V}, \quad \mathbf{D} = \mathcal{A}_-(\delta) + K_-, \quad \frac{H}{2} = \int_{\bar{\lambda}_{g+1}}^{P_\infty^+} d\Omega_2.$

Here $\mathcal{A}_-$ *and* K_- *are the Abel map and Riemann constant with base point* P_∞^-*.*

δ *is a degree-*g *divisor.*

The solution of PLR equation is given by

$$q = 2i\sqrt{\beta}\exp(-isE + itH)\frac{\theta(\mathbf{W} + \mathbf{D} - \mathbf{r})}{\theta(\mathbf{W} + \mathbf{D})}$$

Proof.

To prove this, it suffices to construct a function $\psi = (\psi_1, \psi_2)$ satisfying the following system of linear differential equations:

$$\psi' = \psi L^\lambda, \quad L^\lambda = \frac{1}{2}\begin{pmatrix} i\lambda & q \\ -\bar{q} & -i\lambda \end{pmatrix},$$

$$\dot{\psi} = \psi M^\lambda, \quad M^\lambda = \frac{i}{2\lambda}\begin{pmatrix} -\operatorname{Re}(\dot{q}'/q) & -\dot{q} \\ -\dot{\bar{q}} & \operatorname{Re}(\dot{q}'/q) \end{pmatrix}.$$

For this purpose, we construct functions on the Riemann surface $\mathcal{R}$ satisfying the following conditions:

- ψ_j is meromorphic on $\mathcal{R} \setminus \{P_\infty^\pm, P_0^\pm\}$ with poles on a positive divisor δ of degree g, such that $\dim L(\delta) = 1$ and $\sigma\delta - \delta - P_\infty^- + P_\infty^- \equiv 0 \mod \Gamma$.
- Near $P_\infty^\pm$ and $P_0^\pm$, Φ satisfies:

$$\psi \sim \tfrac{\alpha}{z}\left[(1,0) + O(z)\right] e^{is/(2z)}, \quad P \to P_\infty^+, \quad \alpha \in \mathbb{C},$$

$$\psi \sim \left[(0,1) + O(z)\right] e^{-is/(2z)}, \quad P \to P_\infty^-,$$

$$\psi \sim O(1)\, e^{\pm it/(2\lambda)}, \quad P \to P_0^\pm.$$

Proof(Cont.)

From the conditions on the pole divisor of ψ and the essential singularities near $P_\infty^\pm$ and $P_0^\pm$, it follows that ψ satisfies the above system of linear differential equations. We now proceed to construct ψ explicitly.
A basic property of the theta function is that for a g-degree positive divisor δ with $\dim L(\delta) = 1$, the function

$$\theta(\mathcal{A}_-(P) - \mathcal{A}_-(\delta) - K_-)$$

has zeros at the g points specified by δ. Using this fact together with the Abelian integrals defined earlier, we obtain:

$$\psi_1 = -i\alpha\sqrt{\beta}\,\exp\left[\tfrac{is}{2}\left(\Omega_1 + \tfrac{E}{2}\right) + \tfrac{it}{2}\left(\Omega_2 - \tfrac{H}{2}\right) + \Omega_3\right] \times \frac{\theta(\mathcal{A}_-(P) - \mathbf{W} - \mathbf{D} - \mathbf{r})\,\theta(\mathbf{D} - \mathbf{r})}{\theta(\mathcal{A}_-(P) - \mathbf{D})\,\theta(\mathbf{W} + \mathbf{D})},$$

$$\psi_2 = \exp\left[\tfrac{is}{2}\left(\Omega_1 - \tfrac{E}{2}\right) + \tfrac{it}{2}\left(\Omega_2 + \tfrac{H}{2}\right)\right] \times \frac{\theta(\mathcal{A}_-(P) - \mathbf{W} - \mathbf{D})\,\theta(\mathbf{D})}{\theta(\mathcal{A}_-(P) - \mathbf{D})\,\theta(\mathbf{W} + \mathbf{D})}.$$

Proof(Cont.)

The constant term in ψ_2 after removing the essential singularity at P_∞^- gives the PLR solution q, expressed as:

$$q = 2iA\, e^{-isE+itH}\, \frac{\theta(\mathbf{W}+\mathbf{D}-\mathbf{r})}{\theta(\mathbf{W}+\mathbf{D})}, \quad A = \frac{\theta(\mathbf{D})}{\alpha\, \theta(\mathbf{D}-\mathbf{r})}.$$

From the choice of the Riemann surface and the properties of the anti-holomorphic involution σ, the requirement that the $(2,1)$-component of L^λ equals $-\bar{q}$ implies $A = \sqrt{\beta}$, yielding the explicit expression for q. Finally, the desired ψ can be obtained from the above formulas by removing the (s,t)-independent terms such as $\theta(\mathcal{A}_-(P) - \mathbf{D})$.

Evolution of the Curve

Theorem 7

The curve associated with the Pohlmeyer-Lund-Regge equation evolves via

$$\Psi = \frac{1}{\sqrt{|\psi_1|^2 + |\psi_2|^2}} \begin{pmatrix} \psi_1 & \psi_2 \\ -\bar{\psi}_2 & \bar{\psi}_1 \end{pmatrix}, \quad \gamma = \left(\frac{d}{d\lambda} \Psi \right) \Psi^{-1} \Big|_{\lambda=1}.$$

The components γ_{11}, γ_{21} *are:*

$$\begin{aligned}
\gamma_{11} &= \tfrac{i}{2} \left(\frac{d\Omega_1}{d\lambda} s + \frac{d\Omega_2}{d\lambda} t \right) \\
&\quad + \tfrac{1}{2\rho} \left(|\psi_1|^2 \nabla \log \tfrac{\theta(\mathcal{A}_- - \varphi - r)}{\theta(\mathcal{A}_- + \varphi - r)} + |\psi_2|^2 \nabla \log \tfrac{\theta(\mathcal{A}_- - \varphi)}{\theta(\mathcal{A}_- + \varphi)} \right) \cdot \frac{d\mathcal{A}_-}{d\lambda} \Big|_{\lambda=1}, \\
\gamma_{12} &= \tfrac{\psi_1 \psi_2}{\rho} \left(\nabla \log \tfrac{\theta(\mathcal{A}_- - \varphi)}{\theta(\mathcal{A}_- - \varphi - r)} \cdot \frac{d\mathcal{A}_-}{d\lambda} + \frac{d\Omega_3}{d\lambda} \right) \Big|_{\lambda=1},
\end{aligned}$$

where $\rho = |\psi_1|^2 + |\psi_2|^2$, $\varphi = \mathbf{W} + \mathbf{D}$.

Proof.

We now focus on the case $\lambda = 1$, i.e. $\lambda \in \mathbb{R}$. Denote by $P \in \mathcal{R}$ the point lying above that real value. Because the anti-holomorphic involution σ and the sheet change ι satisfy $\sigma\iota(P) = P$ on the real slice, we immediately obtain,

$$\overline{\Omega_j(P)} = \Omega_j(P) \quad (j = 1, 2, 3),$$

and modulo the period lattice Γ,

$$\overline{\mathcal{A}(P)} = \mathcal{A}(P), \qquad \overline{\mathbf{D}} = -\mathbf{D}, \qquad \bar{\mathbf{r}} = \mathbf{r}.$$

Since each ψ_j is a single-valued function, we may regard the equalities as genuine identities rather than only modulo Γ.

Proof (Cont.)

Compute γ_{11}:

$$\gamma_{11} = -\frac{1}{2}\frac{d}{d\lambda}\log\bigl(|\psi_1|^2+|\psi_2|^2\bigr) + \frac{1}{|\psi_1|^2+|\psi_2|^2}\left(\overline{\psi_1}\,\tfrac{d}{d\lambda}\psi_1 + \overline{\psi_2}\,\tfrac{d}{d\lambda}\psi_2\right)$$

$$= \frac{1}{2}\frac{1}{|\psi_1|^2+|\psi_2|^2}\left(|\psi_1|^2\frac{d}{d\lambda}\log\Bigl(\frac{\psi_1}{\overline{\psi_1}}\Bigr) + |\psi_2|^2\frac{d}{d\lambda}\log\Bigl(\frac{\psi_2}{\overline{\psi_2}}\Bigr)\right).$$

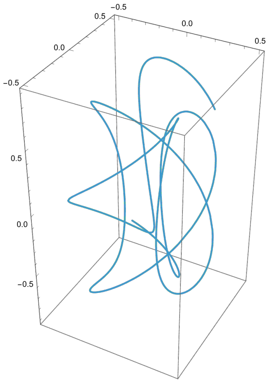
By using the reality relations written above and $\overline{\theta(\mathbf{u})} = \theta(\bar{\mathbf{u}})$, we find

$$\frac{\psi_1}{\overline{\psi_1}} = \exp\bigl(is\,\Omega_1(P) + it\,\Omega_2(P)\bigr)\frac{\theta\bigl(\mathcal{A}_-(P)-\varphi-r\bigr)}{\theta\bigl(\mathcal{A}_-(P)+\varphi-r\bigr)},$$

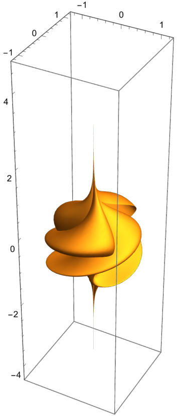
$$\frac{\psi_2}{\overline{\psi_2}} = \exp\bigl(is\,\Omega_1(P) + it\,\Omega_2(P)\bigr)\frac{\theta\bigl(\mathcal{A}_-(P)-\varphi\bigr)}{\theta\bigl(\mathcal{A}_-(P)+\varphi\bigr)},$$

and substituting these expressions yields the desired formula. □

Example



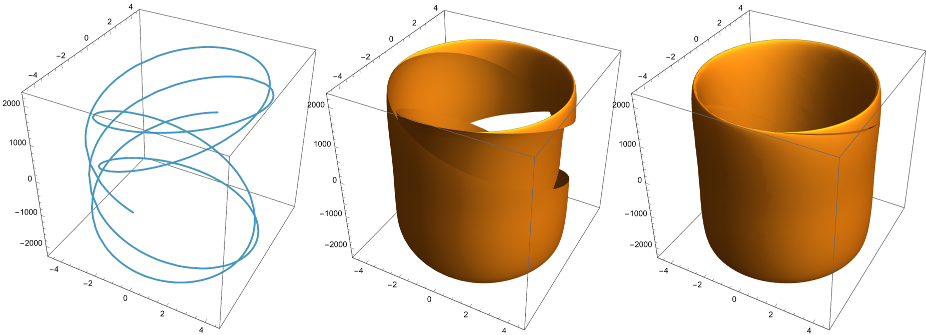
(a) A space curve at time $t = 0, s \in [-5, 5]$



(b) $t \in [-5, 5], s \in [-5, 5]$

Figure: A curve evolution determined from a solution of the PLR equation and its swept surface by the curves.$(\lambda_1 = 3 + i,\ \lambda_2 = 1 + i)$

Example



(a) A space curve at time $t = 0, s \in [-3, 3]$

(b) $t \in [0, 5], s \in [-3, 3]$

(c) $t \in [0, 10], s \in [-3, 3]$

Figure: A curve evolution determined from a solution of the PLR equation and its swept surface by the curves.($\lambda_1 = 1.00345691719 + 0.5i, \ \lambda_2 = -1.00345691719 + 0.5i$)

Forthcoming research

- Closure condition of the curve
- How relate to sine-Gordon equation
- Visualization in terms of higher genus Riemann theta function

References I

E. Date.
Multi-soliton solutions and quasi-periodic solutions of nonlinear equations of sine-Gordon type.
Osaka J. Math. **19** (1982), 125–158.

K. Pohlmeyer
Integrable Hamiltonian systems and iteractions through quadratic constraints.
Comm. Math. Phys. **46**,(1976), 207–221.

F. Lund, T. Regge.
Unified approach to strings and vortices with soliton solutions.
Phys. Rev. D **14**, no. 6 (1978), 1524–1535.

References II

S-P. Kobayashi, Y. Kogo, N. Matsuura
The evolution of a curve induced by the Pohlmeyer-Lund-Regge equation.
J. Nonlinear Sci. **35**, 85 (2025).

Thank you